6

THE FIELD EQUATIONS

The colliding wave problem has been described in terms of two approaching waves in regions II and III, in a background region I, which is here taken to be flat. According to the work of Penrose (1980), the initial data are well set, so that a unique solution exists in the interaction region IV at least in the neighbourhood of the boundaries of regions II and III. It is therefore necessary first to state the relevant field equations in the interaction region, and then to attempt to solve them subject to the appropriate boundary conditions. This formal approach is not always achievable in practice, as analytic solutions in region IV are not easily obtained. It is often more appropriate to start with the field equations for region IV and, once a solution is found, subsequently to consider the particular initial waves that would give rise to it. In either case, it is necessary to formulate the appropriate field equations for the interaction region. The purpose of this chapter is to derive such equations.

Following Szekeres (1972) and Griffiths (1976*b*), the field equations will be derived here using Newman–Penrose techniques.

6.1 The coordinate system

The approaching waves have plane symmetry. On collision they will warp and scatter each other. In the interaction region they will no longer retain the full plane symmetry described by (4.15), though it may be expected that they will retain a two-parameter symmetry group of motions generated by the Killing vectors

$$\xi_1 = \partial_x, \qquad \xi_2 = \partial_y. \tag{6.1}$$

We expect the interaction region to retain plane symmetry in this sense. We therefore initially assume that the Killing vectors ξ_1 and ξ_2 exist throughout the space-time, and that the metric therefore has no explicit dependence on the coordinates x and y in the interaction region. Ultimately of course, this assumption is only justified if it leads to exact solutions.

At each point of the manifold there exist just two null directions orthogonal to the planes spanned by ∂_x and ∂_y. We may now align the two vectors l^μ and n^μ with these two directions. Since the congruences in these directions are hypersurface orthogonal, the vectors l^μ and n^μ are proportional to gradients. It is therefore possible to put

$$l_\mu = A^{-1}u_{,\mu}, \qquad n_\mu = B^{-1}v_{,\mu} \tag{6.2}$$

where A and B are independent of x and y.

We may now choose u and v as suitable null coordinates in the interaction region. At this point, these are arbitrary functions of the coordinates with the same labels in regions II and III continued into the interaction region. It is therefore appropriate to assume that $u \geq 0$ and $v \geq 0$ in region IV. Labelling the coordinates as $(u,\ v,\ x,\ y) = (x^0,\ x^1,\ x^2,\ x^3)$, the null tetrad may now be chosen to be

$$\begin{aligned} l_\mu &= A^{-1}\delta^0_\mu, & l^\mu &= (0, B, Y^2, Y^3) \\ n_\mu &= B^{-1}\delta^1_\mu, & n^\mu &= (A, 0, X^2, X^3) \\ & & m^\mu &= (0, 0, \xi^2, \xi^3) \end{aligned} \tag{6.3}$$

where, putting $i = 2, 3$,

$$X^i = X^i(u,v), \qquad Y^i = Y^i(u,v), \qquad \xi^i = \xi^i(u,v). \tag{6.4}$$

The tetrad and coordinates are now defined up to the following transformations:
(1) Scale transformations or *boosts:*

$$l^{\mu\prime} = \phi l^\mu, \qquad n^{\mu\prime} = \phi^{-1}n^\mu, \qquad \phi = \phi(u,v), \tag{6.5}$$

under which the scale functions A and B transform as

$$A' = \phi^{-1}A, \qquad B' = \phi B.$$

(2) Spatial rotations or *spins:*

$$m^{\mu\prime} = e^{iC}m^\mu, \qquad C = C(u,v). \tag{6.6}$$

This is a rotation in a space-like 2-surface and transforms the ξ^i by

$$\xi^{i\prime} = e^{iC}\xi^i.$$

(3) Relabelling of null hypersurfaces:

$$u' = f(u), \qquad v' = g(v), \tag{6.7}$$

which induces the transformation in the scale functions

$$A' = f_{,u}A \ , \qquad B' = g_{,v}B.$$

(4) Spatial coordinate transformations:

$$u' = u, \qquad v' = v, \qquad x^{i\prime} = x^i + h^i(u,v), \tag{6.8}$$

which transforms the X^i and Y^i components by

$$X^{i\prime} = X^i + h^i_{,u}A \ , \qquad Y^{i\prime} = Y^i + h^i_{,v}B.$$

(5) Linear coordinate transformations:

$$x^{i\prime} = a^i{}_j x^j, \qquad a^i{}_j = \text{consts.} \tag{6.9}$$

6.2 The derivation of the field equations

With the tetrad defined as above, it may be noticed that all components are at most functions of u and v only. It follows that the differential operators, when applied to spin coefficients, take the form

$$D = B\tfrac{\partial}{\partial v}, \qquad \Delta = A\tfrac{\partial}{\partial u}, \qquad \delta = 0. \tag{6.10}$$

The incoming waves have no dependence on the coordinates x and y, and so no such dependence is expected in the interaction region.

The metric equations, which are the commutation relations (2.15) applied to the coordinates, immediately take the form

$$\begin{aligned}
\kappa = \nu = 0, \qquad \rho = \bar\rho, \qquad \mu = \bar\mu& \\
\bar\tau = \pi = 2\bar\beta &= 2\alpha \\
DA &= -(\epsilon + \bar\epsilon)A \\
\Delta B &= (\gamma + \bar\gamma)B \\
\Delta Y^i - DX^i &= (\gamma + \bar\gamma)Y^i + (\epsilon + \bar\epsilon)X^i - 4\alpha\xi^i - 4\bar\alpha\bar\xi^i \\
D\xi^i &= (\rho + \epsilon - \bar\epsilon)\xi^i + \sigma\bar\xi^i \\
\Delta\xi^i &= -(\mu + \bar\gamma - \gamma)\xi^i - \bar\lambda\bar\xi^i.
\end{aligned} \tag{6.11}$$

The first of these equations simply expresses the condition, consistent with (6.2), that the congruences tangent to l^μ and n^μ are geodesic and twist-free. The third and fourth metric equations (6.11) enable the real parts of ϵ and γ to be defined by

$$\epsilon+\bar\epsilon=-B(\log A)_{,v}\ ,\qquad \gamma+\bar\gamma=A(\log B)_{,u}. \tag{6.12}$$

It can thus be seen that the two null congruences do not necessarily have affine parametrization.

Before proceeding to the field equations, it is convenient to introduce modified 'scale-invariant' spin coefficients

$$\begin{array}{lll}
\rho^\circ=\rho B^{-1}, & \sigma^\circ=\sigma B^{-1}, & E^\circ=i(\bar\epsilon-\epsilon)B^{-1},\\
\mu^\circ=\mu A^{-1}, & \lambda^\circ=\lambda A^{-1}, & G^\circ=\ i(\bar\gamma-\gamma)A^{-1},\\
\alpha^\circ=\alpha(AB)^{-1/2}, & Y^{\circ i}=Y^iB^{-1}, & X^{\circ i}=\ X^iA^{-1},\\
\Psi^\circ_0=\Psi_0B^{-2}, & \Psi^\circ_1=\ \Psi_1A^{-1/2}B^{-3/2}, & \Psi^\circ_2=\Psi_2(AB)^{-1},\\
\Psi^\circ_3=\Psi_3B^{-1/2}A^{-3/2}, & \Psi^\circ_4=\ \Psi_4A^{-2}, & \Lambda^\circ=\Lambda(AB)^{-1},\\
\Phi^\circ_{00}=\Phi_{00}B^{-2}, & \Phi^\circ_{01}=\ \Phi_{01}A^{-1/2}B^{-3/2}, & \Phi^\circ_{02}=\Phi_{02}(AB)^{-1},\\
\Phi^\circ_{11}=\Phi_{11}(AB)^{-1}, & \Phi^\circ_{12}=\ \Phi_{12}B^{-1/2}A^{-3/2}, & \Phi^\circ_{22}=\Phi_{22}A^{-2}.
\end{array} \tag{6.13}$$

It is also convenient to introduce the scale-invariant function M, defined by

$$M=\log AB. \tag{6.14}$$

The metric equations (6.11) and the Ricci identities (2.14) now only involve quantities that are invariant with respect to the scale transformations (6.5). Together they give the following:

$$Y^{\circ i}{}_{,u}-X^{\circ i}{}_{,v}=-4(\alpha^\circ\xi^i+\bar\alpha^\circ\bar\xi^i)e^{-M/2} \tag{6.15a}$$

$$\xi^i{}_{,v}=(\rho^\circ+iE^\circ)\xi^i+\sigma^\circ\bar\xi^i \tag{6.15b}$$

$$\xi^i{}_{,u}=-(\mu^\circ-iG^\circ)\xi^i-\bar\lambda\bar\xi^i \tag{6.15c}$$

$$\rho^\circ{}_{,v}=\rho^{\circ 2}+\sigma^\circ\bar\sigma^\circ-\rho^\circ M_{,v}+\Phi^\circ_{00} \tag{6.15d}$$

$$\rho^\circ{}_{,u}=-2\rho^\circ\mu^\circ-4\alpha^\circ\bar\alpha^\circ-\Phi^\circ_{11}-3\Lambda^\circ \tag{6.15e}$$

$$\mu^\circ{}_{,v}=2\rho^\circ\mu^\circ+4\alpha^\circ\bar\alpha^\circ+\Phi^\circ_{11}+3\Lambda^\circ \tag{6.15f}$$

$$\mu^\circ{}_{,u}=-\mu^{\circ 2}-\lambda^\circ\bar\lambda^\circ-\mu^\circ M_{,u}-\Phi^\circ_{22} \tag{6.15g}$$

$$\sigma^\circ{}_{,v}=\sigma^\circ(2\rho^\circ-M^\circ_{,v}+2iE^\circ)+\Psi^\circ_0 \tag{6.15h}$$

$$\sigma^\circ{}_{,u}=-\sigma^\circ(\mu^\circ-2iG^\circ)-\rho^\circ\bar\lambda^\circ-4\bar\alpha^{\circ 2}-\Phi^\circ_{02} \tag{6.15i}$$

$$\lambda^\circ{}_{,v}=\lambda^\circ(\rho^\circ-2iE^\circ)+\mu^\circ\bar\sigma^\circ+4\alpha^{\circ 2}-\Phi^\circ_{20} \tag{6.15j}$$

$$\lambda^\circ{}_{,u}=-\lambda^\circ(2\mu^\circ+M_{,u}+2iG^\circ)-\Psi^\circ_4 \tag{6.15k}$$

$$\alpha^\circ{}_{,v} = \alpha^\circ(3\rho^\circ - \tfrac{1}{2}M_{,v} - iE^\circ) + \bar\alpha^\circ\bar\sigma^\circ + \Phi^\circ_{10} \tag{6.15l}$$

$$\alpha^\circ{}_{,u} = -\alpha^\circ(3\mu^\circ + \tfrac{1}{2}M_{,u} + iG^\circ) - \bar\alpha^\circ\lambda^\circ - \Phi^\circ_{21} \tag{6.15m}$$

$$\tfrac{1}{2}M_{,uv} + \tfrac{1}{2}i(G^\circ{}_{,v} - E^\circ{}_{,u}) = \rho^\circ\mu^\circ - \sigma^\circ\lambda^\circ + 12\alpha^\circ\bar\alpha^\circ + 2\Phi^\circ_{11} \tag{6.15n}$$

$$\Psi^\circ_1 = 2\rho^\circ\bar\alpha^\circ - 2\sigma^\circ\alpha^\circ + \Phi^\circ_{01} \tag{6.15o}$$

$$\Psi^\circ_2 = \rho^\circ\mu^\circ - \sigma^\circ\lambda^\circ + \Phi^\circ_{11} + \Lambda^\circ \tag{6.15p}$$

$$\Psi^\circ_3 = 2\mu^\circ\alpha^\circ - 2\lambda^\circ\bar\alpha^\circ + \Phi^\circ_{21} \tag{6.15q}$$

Following the method of Szekeres, it is convenient at this point to put

$$\begin{aligned} \xi^2 &= \tfrac{1}{\sqrt{2}}e^{(U-V)/2}(\cosh\tfrac{1}{2}W + i\sinh\tfrac{1}{2}W) \\ \xi^3 &= \tfrac{1}{\sqrt{2}}e^{(U+V)/2}(\sinh\tfrac{1}{2}W + i\cosh\tfrac{1}{2}W) \end{aligned} \tag{6.16}$$

where U, V and W are functions of u and v. Using equations (6.15*b,c*) some of the scale-invariant spin coefficients can then be expressed explicitly as

$$\begin{aligned} \rho^\circ &= \tfrac{1}{2}U_v & \mu^\circ &= -\tfrac{1}{2}U_u \\ E^\circ &= -\tfrac{1}{2}V_v\sinh W & G^\circ &= -\tfrac{1}{2}V_u\sinh W \\ \sigma^\circ &= \tfrac{1}{2}iW_v - \tfrac{1}{2}V_v\cosh W & \lambda^\circ &= \tfrac{1}{2}iW_u + \tfrac{1}{2}V_u\cosh W \end{aligned} \tag{6.17}$$

where the suffices u and v refer to partial derivatives taken with respect to these variables. From now on, the commas on metric functions will be omitted and derivatives will be denoted only by suffices.

The method of procedure from this point depends heavily on the form of the Ricci tensor. For colliding gravitational and electromagnetic waves it will be shown that Φ_{01} and Φ_{21} are both zero. In this case it follows from (6.15*l,m*) that, since $\alpha^\circ = 0$ in the initial regions II and III, it is necessary that $\alpha^\circ = 0$ everywhere.

By contrast, for colliding neutrino fields (Griffiths 1976*a*) which will only be considered in Section 20.5, the components Φ_{01} and Φ_{21} are essentially non-zero and the method of integrating the field equations differs significantly. In this, and similar cases, it is clearly not possible to put $\alpha^\circ = 0$ everywhere. Apart from section 20.5, however, the methods for integrating the field equations in these cases will not be pursued.

In this book the emphasis is on gravitational and electromagnetic waves, so only the Einstein–Maxwell equations will be derived explicitly in this section. Einstein's vacuum equations are obtained simply by putting the electromagnetic field components equal to zero.

6.3 The Einstein and Einstein–Maxwell equations

In order to consider the collision of electromagnetic waves, it is convenient first to define the scale-invariant quantities

$$\Phi_0^\circ = \Phi_0 B^{-1}, \qquad \Phi_1^\circ = \Phi_1 (AB)^{-1/2}, \qquad \Phi_2^\circ = \Phi_2 A^{-1}. \tag{6.18}$$

With the conditions (6.11), Maxwell's equations (2.18) now become

$$\begin{aligned} \Phi^\circ_{1,v} &= (2\rho^\circ - \tfrac{1}{2}M_{,v})\Phi_1^\circ \\ \Phi^\circ_{2,v} &= -\lambda^\circ\Phi_0^\circ + 4\alpha^\circ\Phi_1^\circ + (\rho^\circ - iE^\circ)\Phi_2^\circ \\ \Phi^\circ_{0,u} &= -(\mu^\circ - iG^\circ)\Phi_0^\circ - 4\bar{\alpha}^\circ\Phi_1^\circ + \sigma^\circ\Phi_2^\circ \\ \Phi^\circ_{1,u} &= -(2\mu^\circ - \tfrac{1}{2}M_{,u})\Phi_1^\circ. \end{aligned} \tag{6.19}$$

It is immediately clear from the first and last of these equations that, since Φ_1 is zero in regions II and III, it must also be zero in the interaction region. It follows that Φ°_{01} and Φ°_{21} are both zero for colliding electromagnetic waves, as they obviously are for pure gravitational waves. With this condition, equations (6.15 l, m) imply that α° is also zero in region IV, since it is similarly zero in regions II and III.

Equation (6.15a) now implies that, since $\alpha = 0$, it is possible to use a spatial coordinate transformation (6.8) to make both X^i and Y^i zero simultaneously. With this transformation, the metric now takes the form

$$\begin{aligned} \mathrm{d}s^2 = {} & 2e^{-M}\mathrm{d}u\mathrm{d}v \\ & - e^{-U}(e^V\cosh W\mathrm{d}x^2 - 2\sinh W\mathrm{d}x\mathrm{d}y + e^{-V}\cosh W\mathrm{d}y^2). \end{aligned} \tag{6.20}$$

This will be referred to as the Szekeres line element.

The only non-zero spin coefficients are now ρ, σ, ϵ, μ, λ and γ. Apart from the freedom associated with the scale transformation (6.5), these are now expressed explicitly in terms of the metric functions by equations (6.17) and (6.12).

Using (6.17) the remaining components of Maxwell's equations (6.19) now become

$$\begin{aligned} \Phi^\circ_{2,v} &= \tfrac{1}{2}(U_v + iV_v\sinh W)\Phi_2^\circ - \tfrac{1}{2}(iW_u + V_u\cosh W)\Phi_0^\circ \\ \Phi^\circ_{0,u} &= \tfrac{1}{2}(U_u - iV_u\sinh W)\Phi_0^\circ + \tfrac{1}{2}(iW_v - V_v\cosh W)\Phi_2^\circ. \end{aligned} \tag{6.21}$$

The expressions (6.17) may also be substituted back into equations (6.15 d, e, f, g, i, j, n) to give

$$U_{uv} = U_uU_v \tag{6.22a}$$

$$
\begin{aligned}
2U_{vv} &= {U_v}^2 + {W_v}^2 + {V_v}^2\cosh^2 W - 2U_vM_v + 4\Phi^\circ_0\bar{\Phi}^\circ_0 && (6.22b)\\
2U_{uu} &= {U_u}^2 + {W_u}^2 + {V_u}^2\cosh^2 W - 2U_uM_u + 4\Phi^\circ_2\bar{\Phi}^\circ_2 && (6.22c)\\
2V_{uv} &= U_uV_v + U_vV_u - 2(V_uW_v + V_vW_u)\tanh W\\
&\quad + 2(\Phi^\circ_0\bar{\Phi}^\circ_2 + \Phi^\circ_2\bar{\Phi}^\circ_0)\mathrm{sech} W && (6.22d)\\
2W_{uv} &= U_uW_v + U_vW_u + 2V_uV_v\sinh W\cosh W\\
&\quad + 2i(\Phi^\circ_0\bar{\Phi}^\circ_2 - \Phi^\circ_2\bar{\Phi}^\circ_0) && (6.22e)\\
2M_{uv} &= -U_uU_v + W_uW_v + V_uV_v\cosh^2 W. && (6.22f)
\end{aligned}
$$

These are Einstein's field equations written in component form.

The remaining equations (6.15 h, o, p, q, k) give the scale-invariant components of the Weyl tensor in the form

$$
\begin{aligned}
\Psi^\circ_0 &= -\tfrac{1}{2}((V_{vv} - U_vV_v + M_vV_v)\cosh W + 2V_vW_v\sinh W)\\
&\quad + \tfrac{1}{2}i(W_{vv} - U_vW_v + M_vW_v - {V_v}^2\cosh W\sinh W)\\
\Psi^\circ_1 &= 0\\
\Psi^\circ_2 &= \tfrac{1}{2}M_{uv} - \tfrac{1}{4}i(V_uW_v - V_vW_u)\cosh W && (6.23)\\
\Psi^\circ_3 &= 0\\
\Psi^\circ_4 &= -\tfrac{1}{2}((V_{uu} - U_uV_u + M_uV_u)\cosh W + 2V_uW_u\sinh W)\\
&\quad - \tfrac{1}{2}i(W_{uu} - U_uW_u + M_uW_u - {V_u}^2\cosh W\sinh W).
\end{aligned}
$$

It follows from this that it is only the Coulomb component Ψ_2 that appears as an interaction term when the gravitational waves Ψ_0 and Ψ_4 interact.

6.4 Integrating the field equations

Having derived field equations for region IV of the colliding plane wave situation, it is initially necessary to question the possible existence of solutions beyond the initial boundaries with regions II and III. This question has been considered by Yurtsever (1989*a*). In an appendix to this paper, he has given a formal proof of the global existence and uniqueness of solutions in the interaction region as far as the focusing singularity, subject to the boundary conditions which will be discussed in the next chapter.

With this question settled, it is appropriate here to make a few simple observations about the basic approach to the integration of the above field equations.

To start with, it can be seen that equation (6.22*a*) can immediately be integrated to give

$$e^{-U} = f(u) + g(v) \qquad (6.24)$$

where $f(u)$ and $g(v)$ are arbitrary functions that, at this stage, may be assumed to be piecewise C^1.

It may also initially be observed that the Szekeres line element (6.20) is similar to that for the plane wave metric in Rosen form (4.13) that is appropriate for regions II and III, except that the metric functions are here functions of both u and v. In fact it is possible to use (6.20) to describe the space-time in all four regions, where U, V, W and M are functions of u and v in region IV, are functions of u only in region II, are functions of v only in region III, and are constants that can be transformed to be zero in region I. Regions II and III contain approaching plane waves that only differ from those described in Chapter 4 by a simple coordinate transformation (6.7) in u or v.

Using this approach, it is appropriate to choose f and g such that

$$\begin{aligned} f &= \tfrac{1}{2} \qquad \text{for} \qquad u \leq 0 \\ g &= \tfrac{1}{2} \qquad \text{for} \qquad v \leq 0. \end{aligned} \tag{6.25}$$

The approaching waves are thus partially described by the functions $f(u)$ in region II, and $g(v)$ in region III. Of the remaining functions, M can be removed in regions II and III by the transformation (6.7), and V and W must satisfy either (6.22c) or (6.22b). The approaching waves are thus characterized by two independent functions in each region, which together describe the amplitude and polarization of the two approaching waves.

Returning to consider the full field equations in region IV, it may be observed that equations (6.22d,e), together with Maxwell's equations (6.21), are the integrability conditions for the field equations (6.22 b,c,f). Thus if functions U, V, W, Φ_2° and Φ_0° are found to satisfy (6.22a,d,e) and (6.21), then a function M that satisfies the remaining equations (6.22b,c,f) automatically exists. It should not be assumed that this function should be zero. It is not easy to find analytic solutions of equations (6.22d,e) and (6.21) and, once a solution is found, a particular $M(u,v)$ can be obtained by integrating (6.22b,c). In view of the required continuity across the boundaries $u=0$ and $v=0$, non-zero functions $M(u)$ and $M(v)$ will usually occur in regions II and III respectively. Normally, it will be possible to reduce these to zero only in these regions, and then only for purposes of interpreting the solution obtained.

It may be seen from this discussion that, in order to integrate the field equations, attention is focused on equations (6.22 d,e) and (6.21) or, for colliding gravitational waves, just on (6.22 d,e). The emphasis is therefore on these equations as we seek to derive the exact solutions given in subsequent chapters.